TECHNOLOGY REVIEW SOLUTION TO J/A 2 AND J/A 3

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Problem J/A 2:

Jerry Grossman has equipped n children with loaded water pistols and has them standing in an open field with no three of them in a straight line, such that the distances between pairs of them are distinct. At a given signal, each child shoots the closest other child with water. Show that if n is any even number, then it is possible (but not necessarily the case) that every child gets wet. Show that if n is odd, then necessarily at least one child stays dry.

Solution:

Let's tackle the second problem first. If n is odd, let's prove one remains dry via contradiction. Assume all the children get soaked.

If we visualize the children on a plane, and draw an arrow from the child shooting to the child being shot, we notice that for every child to get soaked, closed "arrow cycles" must form. That's because every child must be shot exactly once (if this weren't the case, then there would be a child being shot twice, which leads to the conclusion that there's a child out there that isn't getting shot).

So either we have cycles of two children shooting each other, or bigger cycles of 3 or more children shooting each other.

But the latter is impossible! In any group of 3 or more children, there will always be two which are closest to each other, which means they soak each other. And since no one else can soak them (since no one can be shot more than once), you can't form a cycle of 3 or more.

Hence, we must only have arrow cycles of just pairs of children. But there's the contradiction. We said that n was odd, which means they can't all be in pairs! Hence our original assumption was incorrect, and there is at least one child that doesn't get soaked.

To show the easier case, when n is even, we use the same analysis as above. We saw that if all the children were in pairs that shot each other, then everyone would get wet. So if n is even, it is possible! But that's the only way. In all other cases, the same analysis we did for the odd number of children holds, which shows that at least one child remains dry.

Problem J/A 3:

Each of logicians A, B, and C wears a hat with a positive integer on it. The number on one hat is the sum of the numbers on the other two. The logicians take turns making statements, as follows:

A: "I don't know my number."

B: "My number is 15."

What numbers are on the hats of A and C?

Solution:

The statement made by logician A actually provides a modicum of information to B: the numbers on the hats of B and C are different. (If they were the same, A would know that his hat's number would be the sum of the numbers on B and C's hats.)

With that information, B was able to determine the number on his hat was 15.

B knew that his hat had either the sum *or* the (positive) difference of the numbers on the hats of A and C. And in fact, he must have seen that the number on A was twice the number on C. That's the only way he could have eliminated one of the two possibilities for his hat, because if his hat had the (positive) difference of the the other two hats, his number would have been the same as C! (And logician A's statement told him that that couldn't be the case.) Thus, he was able to deduce his hat's number was the sum of the numbers on A and C's hats.

Since B said his hat had 15 on it, and we know that the number on A was twice the number on C, we know that A and C have 10 and 5 on their hats respectively.